

# Lattice Refining LQC and the Matter Hamiltonian

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In the context of loop quantum cosmology, we parametrise the lattice refinement by a parameter,  $A$ , and the matter Hamiltonian by a parameter,  $\delta$ . We then solve the Hamiltonian constraint for both a self-adjoint, and a non-self-adjoint Hamiltonian operator. Demanding that the solutions for the wave-functions obey certain physical restrictions, we impose constraints on the two-dimensional,  $(A, \delta)$ , parameter space, thereby restricting the types of matter content that can be supported by a particular lattice refinement model.

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## I. INTRODUCTION

Loop quantum gravity is a canonical quantisation of general relativity based on a Hamiltonian formulation with basic variables the connection, which basically carries information about curvature, and the triad, which encodes information about the spatial geometry. Reducing the dynamical variables of the full theory to homogeneous and isotropic models, one gets loop quantum cosmology [1], which is not a field theory.

The fundamental variables of loop quantum cosmology are the holonomies of the  $SU(2)$  connection,  $A_a^i$ , ( $i$  refers to the Lie algebra index and  $a$  is a spatial index with  $a$  and  $i$  taking values 1,2,3) along a given edge, and the corresponding conjugate momentum, which is the flux of the densitised triad,  $E_i^a$ , through a two-surface. Assuming spatially flat, homogeneous and isotropic models, the connection is given by a multiple of the basis one forms, and the triad is obtained from the determinant of the fiducial flat metric,  ${}^0q_{ab}$ , which defines the volume,  $V_0$ , of the elementary cell,  $\mathcal{V}$ . All integrations are performed over the fiducial (elementary) cell,  $\mathcal{V}$ .

To proceed with the quantisation procedure, one has first to construct the Hamiltonian operator. Dynamics are then determined by the Hamiltonian constraint. We emphasise that while in the full theory there are an infinite number of constraints, in the reduced homogeneous and isotropic case there is only one integrated Hamiltonian constraint. Matter is introduced by adding the actions of matter components to the gravitational action. Thus, one just adds the matter contribution to the Hamiltonian constraint. One then obtains difference equations, analogous to the differential Wheeler-DeWitt equations.

Phenomenological reasons [2, 3, 4] require the parameter appearing in the regularisation of the Hamiltonian constraint not to be constant. Considering an underlying lattice which is being refined during dynamical changes of the volume, one allows the number of vertices on the closed loop making up the holonomies to vary dynamically. One has then to implement this requirement in the quantisation procedure of the Hamiltonian constraint.

We parametrise the lattice refinement and the matter Hamiltonian by introducing two parameters,  $A$  and  $\delta$ , respectively. Considering the Hamiltonian operator in the self-adjoint and the non-self-adjoint cases, we solve the constraint equation for both fixed and varying

lattices. Demanding that the solutions should satisfy certain physical requirements, we impose constraints on the two-dimensional,  $(A, \delta)$ , parameter space.

## II. CONSTANT LATTICE

The gravitational part of the Hamiltonian operator,  $\hat{\mathcal{C}}_{\text{grav}}$ , can be written in terms of SU(2) holonomies,  $\hat{h}_i$ , and the triad component,  $p$ , determining in the flat ( $k = 0$ ) model the physical volume of the fiducial cell,  $\mathcal{V}$ , as [2, 5]

$$\hat{\mathcal{C}}_{\text{grav}} = \frac{2i}{\kappa^2 \hbar \gamma^3 \mu_0^3} \text{tr} \sum_{ijk} \epsilon^{ijk} \left( \hat{h}_i^{(\mu_0)} \hat{h}_j^{(\mu_0)} \hat{h}_i^{(\mu_0)-1} \hat{h}_j^{(\mu_0)-1} \hat{h}_k^{(\mu_0)} \left[ \hat{h}_k^{(\mu_0)-1}, \hat{V} \right] \right) \text{sgn}(\hat{p}) , \quad (2.1)$$

where  $\kappa = 8\pi G$ , and  $\hat{V} = \widehat{|p|^{3/2}}$  denotes the volume operator. Note that we use the irreducible representation,  $J = 1/2$ , since in this case the Hamiltonian constraint is free of ill-behaving spurious solutions [2, 7]. The holonomy,  $\hat{h}_i^{(\mu_0)}$ , along the edge parallel to the  $i^{\text{th}}$  basis vector, of length  $\mu_0 V_0^{1/3}$  with respect to the fiducial metric, is [5]

$$\hat{h}_i^{(\mu_0)} = \cos\left(\frac{\mu_0 c}{2}\right) \mathbb{1} + 2 \sin\left(\frac{\mu_0 c}{2}\right) \tau_i , \quad (2.2)$$

where  $\mathbb{1}$  is the identity  $2 \times 2$  matrix and  $\tau_i = -i\sigma_i/2$  is a basis in the Lie algebra SU(2) satisfying the relation

$$\tau_i \tau_j = (1/2) \epsilon_{ijk} \tau^k - (1/4) \delta_{ij} ,$$

with  $\sigma_i$  the Pauli matrices. The pair  $(c, p)$  denotes the coordinates of the two-dimensional gravitational phase-space. The triad component  $p$  determines the physical volume of the fiducial cell and the connection component determines the rate of change of the physical edge length of the fiducial cell. They are related through

$$\{c, p\} = \frac{\kappa \gamma}{3} , \quad (2.3)$$

with  $\gamma$  the Barbero-Immirzi parameter representing a quantum ambiguity parameter of the theory.

The action of the operator  $\exp[\widehat{i(\mu_0 c/2)}]$  on the basis states,  $|\mu\rangle$ , with

$$\hat{p}|\mu\rangle = (\kappa \gamma \hbar |\mu|/6) |\mu\rangle ,$$

where  $\mu$  (a real number) stands for the eigenstates of  $\hat{p}$ , satisfying the orthonormality relation

$$\langle \mu_1 | \mu_2 \rangle = \delta_{\mu_1, \mu_2} ,$$

reads

$$\exp\left[\widehat{\frac{i\mu_0}{2}c}\right] |\mu\rangle = \exp\left[\mu_0 \frac{d}{d\mu}\right] |\mu\rangle = |\mu + \mu_0\rangle ; \quad (2.4)$$

$\mu_0$  is any real number.

The action of the holonomies,  $\hat{h}_i^{(\mu_0)}$ , of the gravitational connection, on the basis states

is given by [5]

$$\hat{h}_i^{(\mu_0)}|\mu\rangle = (\hat{c}\hat{s}\mathbb{1} + 2\hat{s}\hat{n}\tau_i)|\mu\rangle , \quad (2.5)$$

where,

$$\begin{aligned} \hat{c}\hat{s}|\mu\rangle &\equiv \cos(\widehat{\mu_0 c}/2)|\mu\rangle = [|\mu + \mu_0\rangle + |\mu - \mu_0\rangle]/2 , \\ \hat{s}\hat{n}|\mu\rangle &\equiv \sin(\widehat{\mu_0 c}/2)|\mu\rangle = [|\mu + \mu_0\rangle - |\mu - \mu_0\rangle]/(2i) . \end{aligned} \quad (2.6)$$

Thus,

$$\begin{aligned} &\hat{h}_i^{(\mu_0)}\hat{h}_j^{(\mu_0)}\hat{h}_i^{(\mu_0)-1}\hat{h}_j^{(\mu_0)-1}|\mu\rangle \\ &= [(\hat{c}\hat{s}^4 - \hat{s}\hat{n}^4)\mathbb{1} + 2(\mathbb{1} - 4\tau_j\tau_i)\hat{c}\hat{s}^2\hat{s}\hat{n}^2 + 4(\tau_i - \tau_j)\mathbb{1}\hat{c}\hat{s}\hat{s}\hat{n}^3]|\mu\rangle , \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} &\hat{h}_i^{(\mu_0)}[\hat{h}_i^{(\mu_0)-1}, \hat{V}]|\mu\rangle \\ &= (\hat{V} - \hat{c}\hat{s}\hat{V}\hat{c}\hat{s} - \hat{s}\hat{n}\hat{V}\hat{s}\hat{n})\mathbb{1}|\mu\rangle + 2\tau_i(\hat{c}\hat{s}\hat{V}\hat{s}\hat{n} - \hat{s}\hat{n}\hat{V}\hat{c}\hat{s})|\mu\rangle . \end{aligned} \quad (2.8)$$

Substituting Eqs. (2.7) and (2.8) into Eq. (2.1) we obtain

$$\hat{\mathcal{C}}_{\text{grav}}|\mu\rangle = \frac{48i}{\kappa^2\hbar\gamma^3\mu_0^3}\hat{c}\hat{s}^2\hat{s}\hat{n}^2(\hat{s}\hat{n}\hat{V}\hat{c}\hat{s} - \hat{c}\hat{s}\hat{V}\hat{s}\hat{n})|\mu\rangle . \quad (2.9)$$

Using Eq. (2.6) we recover the known expression for the action of the gravitational part of the Hamiltonian constraint, namely<sup>1</sup>

$$\hat{\mathcal{C}}_{\text{grav}}|\mu\rangle = \frac{1}{4}\left(\frac{\hbar}{6\kappa\gamma^3}\right)^{1/2}\mu_0^{-3}S(\mu)[|\mu + 4\mu_0\rangle - 2|\mu\rangle + |\mu - 4\mu_0\rangle] , \quad (2.10)$$

where  $S(\mu)$  is defined by

$$S(\mu) = |\mu + \mu_0|^{3/2} - |\mu - \mu_0|^{3/2} . \quad (2.11)$$

To make the Hamiltonian operator self-adjoint we simply define

$$\hat{\mathcal{H}}_{\text{grav}} = \frac{1}{2}(\hat{\mathcal{C}}_{\text{grav}} + \hat{\mathcal{C}}_{\text{grav}}^\dagger) , \quad (2.12)$$

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<sup>1</sup> Being interested in the large scale behaviour of the loop quantum cosmology equations, we neglect the sign ambiguity that arises from the two different orientations of the triad.

which acts on the basis states as

$$\begin{aligned} \hat{\mathcal{H}}_{\text{grav}}|\mu\rangle = & \frac{1}{8} \left( \frac{\hbar}{6\kappa\gamma^3} \right)^{1/2} \mu_0^{-3} \left( \left[ S(\mu) + S(\mu + 4\mu_0) \right] |\mu + 4\mu_0\rangle \right. \\ & \left. - 4S(\mu)|\mu\rangle + \left[ S(\mu) + S(\mu - 4\mu_0) \right] |\mu - 4\mu_0\rangle \right). \end{aligned} \quad (2.13)$$

Taking the continuum limit ( $\mu \gg \mu_0$ ) of the Hamiltonian constraint equation

$$\hat{\mathcal{H}}_{\text{grav}}|\Psi\rangle = -\hat{\mathcal{H}}_\phi|\Psi\rangle, \quad (2.14)$$

and expanding the general state  $|\Psi\rangle$  in the kinematical Hilbert space in terms of the basis states,  $|\mu\rangle$ , as

$$|\Psi\rangle = \sum_{\mu} \Psi_{\mu}(\phi) |\mu\rangle, \quad (2.15)$$

where the coefficients  $\Psi_{\mu}$  are not continuous with respect to  $\mu$  and the dependence of the coefficients on  $\phi$  represents the matter degrees of freedom, we get

$$\begin{aligned} -\mathcal{H}_{\phi}\Psi_{\mu} = & \frac{3}{8} \left( \frac{\hbar}{6\kappa\gamma^3} \right)^{1/2} \mu_0^{-2} \mu^{1/2} \left[ 2(\Psi_{\mu+4\mu_0} - 2\Psi_{\mu} + \Psi_{\mu-4\mu_0}) \right. \\ & - \frac{2\mu_0}{\mu} (\Psi_{\mu-4\mu_0} - \Psi_{\mu+4\mu_0}) - \frac{2\mu_0^2}{\mu^2} (\Psi_{\mu-4\mu_0} + \Psi_{\mu+4\mu_0}) \\ & \left. - \frac{\mu_0^2}{12\mu^2} (\Psi_{\mu+4\mu_0} - 2\Psi_{\mu} + \Psi_{\mu-4\mu_0}) + \mathcal{O}(\mu_0^3) \right], \end{aligned} \quad (2.16)$$

where  $\hat{\mathcal{H}}_{\phi}|\Psi\rangle = \mathcal{H}_{\phi}|\Psi\rangle$  is assumed to act diagonally on the basis states  $|\mu\rangle$ . We note that the kinematical inner product of the general states reads

$$\langle\Psi|\Psi'\rangle = \sum_{\mu} \bar{\Psi}_{\mu} \Psi'_{\mu}. \quad (2.17)$$

with the requirement that a state in the kinematical Hilbert space must have a finite kinematical norm. The basis states  $|\mu\rangle$  are eigenstates of the volume operator and while the eigenvalues  $\mu$  are valued on the whole real line, the states are normalisable with respect to the kinematical inner product [2].

Assuming that the wave-function does not vary much on scales smaller than  $4\mu_0$  (known as *pre-classicality* [8]), one can approximate  $\Psi_{\mu}(\phi)$  as  $\Psi_{\mu}(\phi) \approx \Psi(\mu, \phi)$ . Then using Taylor expansion the constraint equation reduces to the Wheeler-DeWitt equation [9]

$$-\mathcal{H}_{\phi}\Psi(\mu, \phi) = 6 \left( \frac{\hbar}{6\kappa\gamma^3} \right)^{1/2} \left[ \frac{\partial^2}{\partial \mu^2} (\mu^{1/2} \Psi(\mu, \phi)) + \mu^{1/2} \frac{\partial^2 \Psi(\mu, \phi)}{\partial \mu^2} + \mathcal{O}(\mu_0) \right]; \quad (2.18)$$

we have re-introduced the dependence on the matter degrees of freedom  $\phi$ . The quantum constraint, which is a difference rather than a differential equation, constraints the coefficients  $\Psi(\mu, \phi)$  to ensure that  $|\Psi\rangle$  is a physical state.

The non-self-adjoint version of the operator produces a different factor ordering, namely

$$-\mathcal{H}_\phi \Psi(\mu, \phi) = 12 \left( \frac{\hbar}{6\kappa\gamma^3} \right)^{1/2} \left[ \frac{\partial^2}{\partial \mu^2} (\mu^{1/2} \Psi(\mu, \phi)) + \mathcal{O}(\mu_0) \right], \quad (2.19)$$

which affects the conditions on normalisability to be discussed later.

### III. LATTICE REFINEMENT

The case of a dynamically altering holonomy length scale,  $\tilde{\mu}(\mu)$ , is required for several phenomenological reasons [2, 3, 4]. However, this is not just a naive substitution,  $\mu_0 \rightarrow \tilde{\mu}(\mu)$ , in the previous equations. One can immediately realise that this would lead to difficulties, since there would be extra terms arising in Eq. (2.18) as a result of the dynamics of the underlying grid.

To derive the correct constraint equation we need to introduce the varying length scale into the definition of the holonomies [5]

$$\hat{h}_i = \exp \left[ \widehat{\frac{-i\sigma_i}{2} \tilde{\mu} c} \right], \quad (3.1)$$

where the reader should keep in mind that  $\tilde{\mu}$  depends on  $\mu$ . Geometric considerations [5] imply that, after quantising,

$$\exp \left[ \widehat{\frac{-i\sigma_i}{2} \tilde{\mu} c} \right] |\Psi(\mu, \phi)\rangle = \exp \left[ \tilde{\mu} \frac{\partial}{\partial \mu} \right] |\Psi(\mu, \phi)\rangle. \quad (3.2)$$

This however is no longer a simple shift operator, since  $\tilde{\mu}$  is a function of  $\mu$ . Consider changing the representation from  $\mu$  to

$$\nu = \tilde{\mu}_0 \int \frac{d\mu}{\tilde{\mu}(\mu)}, \quad (3.3)$$

where  $\tilde{\mu}_0$  is a constant. In this representation we have

$$\hat{h}_i |\nu\rangle = \exp \left[ \tilde{\mu}(\mu) \frac{d}{d\mu} \right] |\nu\rangle = \exp \left[ \tilde{\mu}_0 \frac{d}{d\nu} \right] |\nu\rangle = |\nu + \tilde{\mu}_0\rangle. \quad (3.4)$$

We can then proceed as before and define

$$\begin{aligned} \widehat{\text{sn}} |\nu\rangle &\equiv \sin \left[ \widehat{\frac{\tilde{\mu} c}{2}} \right] |\nu\rangle = \frac{1}{2i} \left[ |\nu + \tilde{\mu}_0\rangle - |\nu - \tilde{\mu}_0\rangle \right], \\ \widehat{\text{cs}} |\nu\rangle &\equiv \cos \left[ \widehat{\frac{\tilde{\mu} c}{2}} \right] |\nu\rangle = \frac{1}{2} \left[ |\nu + \tilde{\mu}_0\rangle + |\nu - \tilde{\mu}_0\rangle \right]. \end{aligned}$$

There is however a problem in defining the volume eigenvalue, since this requires an explicit

relation between  $\nu$  and  $\mu$  given by  $\tilde{\mu}$ . Assuming

$$\tilde{\mu} = \mu_0 \mu^A , \quad (3.5)$$

one has

$$\nu = \frac{\tilde{\mu}_0 \mu^{1-A}}{\mu_0 (1-A)} \quad (3.6)$$

(up to a constant that can be set equal to 0), leading to

$$\begin{aligned} \hat{V}|\nu\rangle &= \left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2} \mu^{3/2}|\nu\rangle \\ &= \left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2} \left[\frac{\mu_0(1-A)}{\tilde{\mu}_0}\right]^{3/2(1-A)} \nu^{3/2(1-A)}|\nu\rangle . \end{aligned} \quad (3.7)$$

### A. Non-self-adjoint case

Let us calculate the action of Eq. (2.1) on the basis state  $|\nu\rangle$ :

$$\hat{\mathcal{C}}_{\text{grav}}|\nu\rangle = \frac{1}{4\mu_0^3} \left(\frac{\hbar}{6\kappa\gamma^3}\right)^{1/2} (\alpha\nu)^{3A/(A-1)} S(\nu) \left(|\nu + 4\tilde{\mu}_0\rangle - 2|\nu\rangle + |\nu - 4\tilde{\mu}_0\rangle\right) , \quad (3.8)$$

where

$$\alpha = \mu_0(1-A)/\tilde{\mu}_0 , \quad (3.9)$$

and  $S(\nu)$  is defined by

$$S(\nu) = \left[(\nu + \tilde{\mu}_0)\alpha\right]^{3/2(1-A)} - \left[(\nu - \tilde{\mu}_0)\alpha\right]^{3/2(1-A)} . \quad (3.10)$$

One can easily check that  $A = 0$  reproduces Eq. (2.10), if it is taken that  $\tilde{\mu}_0 = \mu_0$ . After a long but straight forward expansion in the  $\nu \gg \tilde{\mu}_0$  limit and under the assumption of pre-classicality, one finds

$$\begin{aligned} \hat{\mathcal{C}}_{\text{grav}}|\Psi(\nu, \phi)\rangle &= \sum_{\nu} 12(1-A)^2 \left(\frac{\hbar}{6\kappa\gamma^3}\right)^{1/2} \alpha^{-3/2(1-A)} \nu^{(1-4A)/2(1-A)} \\ &\times \left[ \frac{\partial^2 \Psi(\nu, \phi)}{\partial \nu^2} + \frac{1-4A}{(1-A)\nu} \frac{\partial \Psi(\nu, \phi)}{\partial \nu} + \frac{(1+2A)(4A-1)}{4(1-A)^2 \nu^2} \Psi(\nu, \phi) \right. \\ &\quad \left. + \mathcal{O}(\tilde{\mu}_0) \right] |\nu\rangle . \end{aligned} \quad (3.11)$$

For a non-self-adjoint Hamiltonian operator, the Hamiltonian constraint equation reads

$$\frac{\partial^2 \Psi(\nu, \phi)}{\partial \nu^2} + \frac{B}{\nu} \frac{\partial \Psi(\nu, \phi)}{\partial \nu} + \frac{C}{\nu^2} \Psi(\nu, \phi) + \beta \mathcal{H}_{\phi} \nu^{-B/2} \Psi(\nu, \phi) + \mathcal{O}(\tilde{\mu}_0) = 0 , \quad (3.12)$$

where

$$\begin{aligned} B &= \frac{1-4A}{(1-A)} \\ C &= \frac{(1+2A)(4A-1)}{4(1-A)^2} \\ \beta &= \frac{\alpha^{3/2/(1-A)}}{12(1-A)^2} \left( \frac{6\kappa\gamma^3}{\hbar} \right)^{1/2}. \end{aligned} \quad (3.13)$$

We note that for a fixed lattice,  $A, B, C$  and  $\beta$  are given by

$$A = 0 \quad , \quad B = 1 \quad , \quad C = -1/4 \quad , \quad \beta = (6\kappa\gamma^3/\hbar)^{1/2} (\mu_0/\tilde{\mu}_0)^{3/2} / 12. \quad (3.14)$$

Considering lattice refinement in the case of a non-self-adjoint Hamiltonian operator, one has  $\nu = \tilde{\mu}_0\mu/\mu_0$ . Thus, by keeping  $\tilde{\mu}_0$  general all we have done is to re-scale  $\mu$ . Setting  $\mu_0 = \tilde{\mu}_0$  we get back Eq. (2.19).

The specific lattice refinement  $A = -1/2$  [3, 5] is clearly a particularly fortitious choice as it results in  $C = 0$ . Notice however that choosing  $A = 1/4$  results in a further simplification, leading to

$$\frac{\partial^2 \Psi(\nu, \phi)}{\partial \nu^2} + \frac{1}{12} \left( \frac{6\kappa\gamma^3}{\hbar} \right)^{1/2} \mathcal{H}_\phi \Psi(\nu, \phi) = 0, \quad (3.15)$$

assuming  $\tilde{\mu}_0 = \mu_0$ . This unphysical lattice refinement choice results in dynamics that are well approximated by the Wheeler-DeWitt equation in the large scale limit (for slowly varying wave-functions). This remark highlights the importance of understanding the origin of the lattice refinement in the full theory. Unfortunately, at present there is little theoretical reason for discounting such physically unacceptable scenarios, and one must rely on his/her phenomenological intuition.

## B. Self-adjoint case

Let us repeat the above procedure for the case of a self-adjoint Hamiltonian operator,

$$\hat{\mathcal{H}}_{\text{grav}} = (\hat{\mathcal{C}}_{\text{grav}} + \hat{\mathcal{C}}_{\text{grav}}^\dagger)/2.$$

Acting on the state  $|\nu\rangle$  one has

$$\begin{aligned} \hat{\mathcal{H}}_{\text{grav}}|\nu\rangle &= \frac{1}{8\mu_0^3} \left( \frac{\hbar}{6\kappa\gamma^3} \right)^{1/2} (\alpha\nu)^{3A/(A-1)} \left[ \left\{ S(\nu + 4\tilde{\mu}_0) + S(\nu) \right\} |\nu + 4\tilde{\mu}_0\rangle \right. \\ &\quad \left. - 4S(\nu) + \left\{ S(\nu - 4\tilde{\mu}_0) + S(\nu) \right\} |\nu - 4\tilde{\mu}_0\rangle \right]. \end{aligned} \quad (3.16)$$

Expanding  $\hat{\mathcal{H}}_{\text{grav}}|\Psi\rangle = -\hat{\mathcal{H}}_\phi|\Psi\rangle$ , one obtains

$$\frac{\partial^2 \Psi(\nu, \phi)}{\partial \nu^2} + \frac{\tilde{B}}{2\nu} \frac{\partial \Psi(\nu, \phi)}{\partial \nu} + \tilde{C} \nu^{-2} \Psi(\nu, \phi) + \beta \mathcal{H}_\phi \nu^{-B/2} \Psi(\nu, \phi) + \mathcal{O}(\tilde{\mu}_0) = 0, \quad (3.17)$$

where  $B$  and  $\beta$  are given by Eq. (3.13a) and Eq. (3.13c), respectively, and

$$\begin{aligned}\tilde{B} &= \frac{1 - 10A}{1 - A} , \\ \tilde{C} &= \frac{(1 + 2A)(4A - 1) + 12A(2A - 1)}{8(1 - A)^2} .\end{aligned}\tag{3.18}$$

As expected, setting  $A = 0$  and  $\tilde{\mu}_0 = \mu_0$  gives back Eq. (2.18). Once again we see that the choice  $A = -1/2$  [3, 5] produces a particular simplification.

### C. Physical sector

In general, not all solutions to the quantum constraint equation, Eq. (3.17) in the case of a self-adjoint Hamiltonian operator, are normalisable with respect to the physical inner product [5, 6]. In what follows, we are only interested in physical states. The physical Hilbert space consists of solutions to the quantum constraint equation which have finite norm with respect to the physical inner product. The inner product on physical states can be obtained by requiring that real classical observables be represented on the physical Hilbert space by self-adjoint operators [6]. The physical inner product has been calculated [5, 6] if the only matter source is a massless scalar field. Following the same procedure, we will compute the inner product of physical states for the model we are considering here.

The (total) Wheeler-DeWitt constraint equation reads

$$\left(\hat{\mathcal{H}}_{\text{grav}} + \hat{\mathcal{H}}_{\phi}\right) \Psi = 0 .\tag{3.19}$$

Since we are interested in the large scale limit, we approximate the matter Hamiltonian,  $\hat{\mathcal{H}}_{\phi}$ , with  $\hat{\mathcal{H}}_{\phi} = \hat{\nu}^{\delta} \hat{\epsilon}(\phi)$  (the reader is referred to the next Section). Thus,

$$\hat{\epsilon}(\phi) \Psi \equiv \epsilon(\phi) \Psi = -\nu^{-\delta} \hat{\mathcal{H}}_{\text{grav}} \Psi .\tag{3.20}$$

In the classical theory,  $\epsilon(\phi)$  is a Dirac observable since it is a constant of motion [5]. Even though  $\nu(\phi)$  is not a constant of motion, assuming that  $\nu(\phi)$  is a monotonic function (with respect to  $\phi$ ), then  $\nu(\phi_0)$  is a Dirac observable for any fixed  $\phi_0$  [5]. Modulo an overall scaling, the unique inner product which makes these operators self-adjoint is [5]

$$\langle \Psi_1 | \Psi_2 \rangle_{\text{phys}} = \int_{\phi=\phi_0} d\nu |\nu|^{\delta} \bar{\Psi}_1 \Psi_2 .\tag{3.21}$$

The finite norm of the physical wave-functions, defined by Eq. (3.21), is conserved, *i.e.*, independent of the choice of  $\phi = \phi_0$ . From Eq. (3.21) one concludes that the solutions of the constraint are normalisable provided they decay, on large scales, faster than  $\nu^{-1/(2\delta)}$ . One arrives to the same conclusion for the case of a constant lattice, with  $\nu$  replaced by  $\mu$ .

It is important to note that, in general the approximation of  $\hat{\mathcal{H}}_{\phi} = \hat{\nu}^{\delta} \hat{\epsilon}(\phi)$  is only valid on the large scale  $\nu$  limit, implying that the integrand of Eq. (3.21) is only valid for  $\nu \gg 1$ . However, it is certainly *necessary* that the large scale behaviour of the wave-functions be normalisable with respect to Eq. (3.21). Thus, the constraint we have found is a *necessary* but not *sufficient* condition for the wave-functions to be considered physical.



#### IV. SOLVING THE CONSTRAINT EQUATION

To solve the constraint equation one needs to know the specific form of  $\mathcal{H}_\phi$ . In general,  $\mathcal{H}_\phi$  has two terms with different scale dependence, however since we are concerned only with the large scale limit, one of these terms will be the dominant one. Making this approximation, one can write

$$\beta\mathcal{H}_\phi = \epsilon_\mu(\phi)\mu^{\delta_\mu} \quad \text{or} \quad \beta\mathcal{H}_\phi = \epsilon_\nu(\phi)\nu^{\delta_\nu}, \quad (4.1)$$

where the functions  $\epsilon_\mu$ ,  $\epsilon_\nu$  are constant with respect to  $\mu$ ,  $\nu$ , respectively. The general analytical solutions read

$$\begin{aligned} \text{non-self-adjoint, fixed lattice } \Psi(\mu) &= C_1 J_{2/(3+2\delta_\mu)} \left( \frac{4\sqrt{\epsilon_\mu}}{3+2\delta_\mu} \mu^{(3+2\delta_\mu)/4} \right) \\ &\quad + C_2 Y_{2/(3+2\delta_\mu)} \left( \frac{4\sqrt{\epsilon_\mu}}{3+2\delta_\mu} \mu^{(3+2\delta_\mu)/4} \right) \\ \text{self-adjoint, fixed lattice } \Psi(\mu) &= C_1 \mu^{1/4} J_{\sqrt{3}/(3+2\delta_\mu)} \left( \frac{4\sqrt{\epsilon_\mu}}{3+2\delta_\mu} \mu^{(3+2\delta_\mu)/4} \right) \\ &\quad + C_2 \mu^{1/4} Y_{\sqrt{3}/(3+2\delta_\mu)} \left( \frac{4\sqrt{\epsilon_\mu}}{3+2\delta_\mu} \mu^{(3+2\delta_\mu)/4} \right) \\ \text{non-self-adjoint, varying lattice } \Psi(\nu) &= C_1 \nu^{-3A/2/(A-1)} J_{(2x)^{-1}} \left( \frac{\sqrt{\epsilon_\nu}}{x} \nu^x \right) \\ &\quad + C_2 \nu^{-3A/2/(A-1)} Y_{(2x)^{-1}} \left( \frac{\sqrt{\epsilon_\nu}}{x} \nu^x \right) \\ \text{self-adjoint, varying lattice } \Psi(\nu) &= C_1 \nu^{(1+8A)/4/(1-A)} J_{x^{-1}y} \left( \frac{\sqrt{\epsilon_\nu}}{x} \nu^x \right) \\ &\quad + C_2 \nu^{(1+8A)/4/(1-A)} Y_{x^{-1}y} \left( \frac{\sqrt{\epsilon_\nu}}{x} \nu^x \right), \end{aligned} \quad (4.2)$$

with

$$\begin{aligned} x &= \frac{2\delta_\nu(1-A)+3}{4(1-A)} \\ y &= \frac{\sqrt{3(12A+1)}}{4(1-A)}; \end{aligned} \quad (4.3)$$

$J$  and  $Y$  are Bessel functions of the first and second kind, respectively, and  $C_1, C_2$  are integration constants. Note that we suppressed the  $\phi$  dependence for clarity. We wrote explicitly the solutions for the non-self-adjoint, as well the self-adjoint case for both a fixed and a varying lattice. In particular, for the *physically* justified choice  $A = -1/2$  [3, 5] the solution of the Hamiltonian constraint equation, in the case of a non-self-adjoint Hamiltonian operator and a varying lattice, reads

$$\Psi(\nu) = C_1 \nu^{-1/2} J_{\frac{i\sqrt{15}}{3(\delta_\nu+1)}} \left( \frac{2\sqrt{\epsilon_\nu}}{\delta_\nu+1} \nu^{(\delta_\nu+1)/2} \right) + C_2 \nu^{-1/2} Y_{\frac{i\sqrt{15}}{3(\delta_\nu+1)}} \left( \frac{2\sqrt{\epsilon_\nu}}{\delta_\nu+1} \nu^{(\delta_\nu+1)/2} \right). \quad (4.4)$$

Among such solutions, we will only consider the physical ones. This immediately eliminates the solutions to the non-self-adjoint Hamiltonian constraint, since it is not possible to find self-adjoint Dirac observables for these cases. We nevertheless find interesting to compare the non-self-adjoint solutions to the self-adjoint ones and we thus apply the norm defined by Eq. (3.21) to both (self-adjoint and non-self-adjoint) sets of solutions. This is done simply to complete the formal comparison between the two cases and it is to be remembered that the normalisation constraint produced here is only rigorous for the self-adjoint case.

Using the asymptotic expansions of the Bessel functions,

$$\begin{aligned}\lim_{z \rightarrow \infty} J_\beta(z) &\rightarrow \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \\ \lim_{z \rightarrow \infty} Y_\beta(z) &\rightarrow \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\beta\pi}{2} - \frac{\pi}{4}\right),\end{aligned}$$

we find that the solutions oscillate within an envelope that scales as

$$\begin{aligned}\text{non-self-adjoint, fixed lattice } \Psi(\mu) &\propto \mu^{-(3+2\delta_\mu)/8} \\ \text{self-adjoint, fixed lattice } \Psi(\mu) &\propto \mu^{-(1+2\delta_\mu)/8} \\ \text{non-self-adjoint, varying lattice } \Psi(\nu) &\propto \nu^{[12A-2\delta_\nu(1-A)-3]/8/(1-A)} \\ \text{self-adjoint, varying lattice } \Psi(\nu) &\propto \nu^{[16A-1-2\delta_\nu(1-A)]/8/(1-A)}.\end{aligned}\tag{4.5}$$

As shown in Section III C if the solutions are to be normalisable,  $\Psi(\nu)$  must not grow faster than  $\Psi(\nu) \propto \nu^{-1/(2\delta)}$ , which imposes constraints on the scale dependence of the allowed matter component. More precisely,

$$\begin{aligned}-\frac{3}{4} &< \frac{\delta_\mu}{2} - \frac{1}{\delta_\mu} && \text{non-self-adjoint, fixed lattice} \\ -\frac{1}{4} &< \frac{\delta_\mu}{2} - \frac{1}{\delta_\mu} && \text{self-adjoint, fixed lattice} \\ \frac{12A-3}{4(1-A)} &< \frac{\delta_\nu}{2} - \frac{1}{\delta_\nu} && \text{non-self-adjoint, varying lattice} \\ \frac{16A-1}{4(1-A)} &< \frac{\delta_\nu}{2} - \frac{1}{\delta_\nu} && \text{self-adjoint, varying lattice}.\end{aligned}\tag{4.6}$$

In the self-adjoint and non-self-adjoint Hamiltonian operator cases with lattice refinement, the growth is taken w.r.t.  $\nu$ . If we require the semi-classical wave-functions not to grow w.r.t.  $\mu$ , there is an additional constraint, namely  $A < 1$ , to ensure that increasing  $\mu$  corresponds to increasing  $\nu$ .

One should also keep in mind that since the solutions are only valid on large scales, one must ensure that the large argument expansions of the Bessel functions apply in this limit. The expansions are valid for

$$\delta_\mu > -3/2,\tag{4.7}$$

in both cases of a self-adjoint and a non-self-adjoint Hamiltonian operator considering a fixed lattice, and for

$$\delta_\nu > \frac{3}{2(A-1)},\tag{4.8}$$

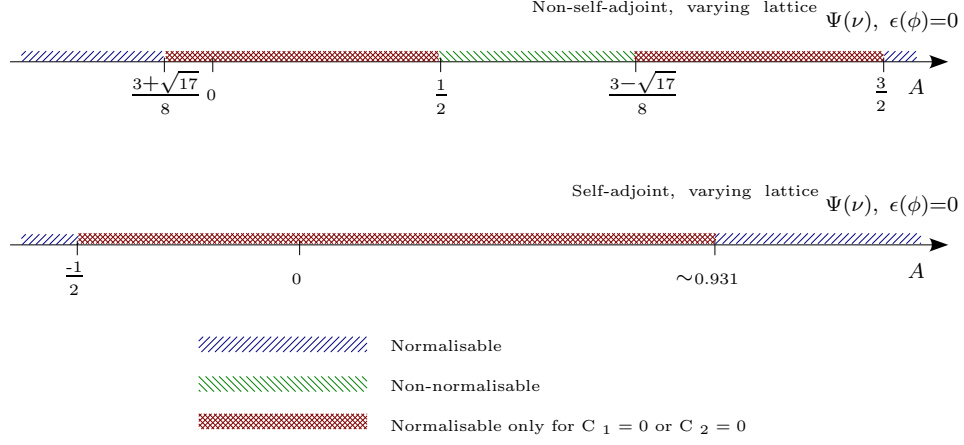


FIG. 1: Regions in which the solutions to the different vacuum Hamiltonian constraint equations are normalisable at large scales. Note that self-adjoint, varying lattice  $\Psi(\nu)$  is oscillatory only for  $A < -1/12$ .

in both cases of a self-adjoint and a non-self-adjoint Hamiltonian operator considering a varying lattice. Beyond these limits the wave-functions, on large scales decay like  $1/y$ , where the argument of the Bessel functions is  $\mu^y$ , or  $\nu^y$ , respectively. Whilst these wave-function may be normalisable, they lack a semi-classical interpretation and hence would not produce classical cosmology at large scales [10]. The regions where the different wave-function coefficients are bounded are shown in Fig. 2 along with the limit of the expansion.

A particularly interesting case is that of the vacuum, where  $\mathcal{H}_\phi = 0$ . This corresponds to  $\epsilon(\phi) = 0$  or  $\delta_\nu = -\infty$ , which makes the norm calculated in Section III C trivial. In this case the norm can be taken to be,

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{\phi=\phi_0} d\nu \bar{\Psi}_1 \Psi_2, \quad (4.9)$$

and correspondingly for  $\mu$ . The solutions to the four constraint equations are

$$\begin{aligned}
 \text{non-self-adjoint, fixed lattice } \Psi(\mu) &= C_1 \mu^{1/2} + C_2 \mu^{-1/2} \\
 \text{self-adjoint, fixed lattice } \Psi(\mu) &= C_1 \mu^{(1+\sqrt{3})/4} + C_2 \mu^{(1-\sqrt{3})/4} \\
 \text{non-self-adjoint, varying lattice } \Psi(\nu) &= \tilde{C}_1 \nu^{(4A-1)/2} + \tilde{C}_2 \nu^{(2A+1)/2} \\
 \text{self-adjoint, varying lattice } \Psi(\nu) &= \tilde{C}_1 \nu^{(8A+1+\sqrt{36A+3})/4} + \tilde{C}_2 \nu^{(8A+1-\sqrt{36A+3})/4}. \quad (4.10)
 \end{aligned}$$

Clearly the two cases of Hamiltonian operator for a fixed lattice are bounded only for specific choices of the integration constants, which amounts to special initial conditions. For the lattice refinement case however, there are several regions in which the solutions are bounded, shown in Fig. 1. It is also worth noticing that only the self-adjoint lattice refinement case produces oscillatory solutions (for  $A < -1/12$ ) and hence have a simple semi-classical dynamical interpretation [10].

## V. LARGE SCALE CLASSICAL BREAKDOWN

The form of the wave-functions indicates that the period of oscillations can decrease as the scale increases, which implies that at sufficiently large scales the assumption that the wave-functions are pre-classical may break down [10, 11]. This would then lead to quantum gravity corrections at large scale (classical) physics. The need to avoid this undesired event, was indeed one of the motivations behind lattice refinement. However, the level of lattice refinement necessary depends on the matter content.

One way to investigate this is to look at the separation between the zeros of the wave-functions [4] (for an alternative method see Ref. [3]). For the non-self adjoint Hamiltonian case we find that the  $n^{\text{th}}$  zero (for large scales) occurs at

$$\nu_n = \left[ \frac{\pi (2\delta_\nu(A-a) - 3)}{4(A-1)\sqrt{\epsilon_\nu}} \right]^{4(A-1)/[2\delta_\nu(A-1)-3]} (n+C)^{4(A-1)/[2\delta_\nu(A-1)-3]} , \quad (5.1)$$

where

$$C = \frac{1}{\pi} \tan^{-1} \left( -\frac{C_1}{C_2} \right) + \frac{(A-1)}{2\delta_\nu(A-1)-3} \pm \frac{1}{2} + \frac{1}{4} , \quad (5.2)$$

is a constant. Since Eq. (5.1) is derived from the large argument expansion of Eq. (4.2), it is only valid for  $\delta_\nu > 3/2/(A-1)$ . Using a Taylor expansion we find

$$\lim_{\text{large } \nu} \Delta\nu_n = \frac{\pi}{\sqrt{\epsilon_\nu}} \nu^{\frac{(4-2\delta_\nu)(1-A)-3}{4(1-A)}} + \mathcal{O} \left( \nu^{\frac{4(1-\delta_\nu)(1-A)-6}{4(1-A)}} \right) . \quad (5.3)$$

Note that the Taylor expansion is valid for  $\delta > 1 + 3/2/(A-1)$ . Using

$$\nu = \tilde{\mu}_0 \frac{\mu^{1-A}}{\mu_0(1-A)} ,$$

we find

$$\lim_{\text{large } \nu} \Delta\nu_n \propto \mu_n^{(\delta_\nu-2)(A-1)/2-3/4} + \mathcal{O} \left( \mu_n^{(\delta_\nu-1)(A-1)-3/2} \right) . \quad (5.4)$$

The lattice refinement will support all oscillations of the wave-function, provided  $\Delta\nu_n$  is larger than  $\nu_c$ , the scale at which the underlying discreteness becomes important. The condition for the continuum limit to be valid, is that the wave-function must vary slowly on scales of the order of [4]

$$\mu_c = 4\tilde{\mu} , \quad (5.5)$$

or, equivalently,

$$\mu_c = 4\mu_0\mu^A . \quad (5.6)$$

Then from Eq. (3.6) one gets

$$\nu_c = \frac{\tilde{\mu}_0(4\mu^A)^{1-A}}{\mu_0^A(1-A)} . \quad (5.7)$$

Equations (5.4) and (5.7) imply that lattice refinement will be sufficient to prevent quantum

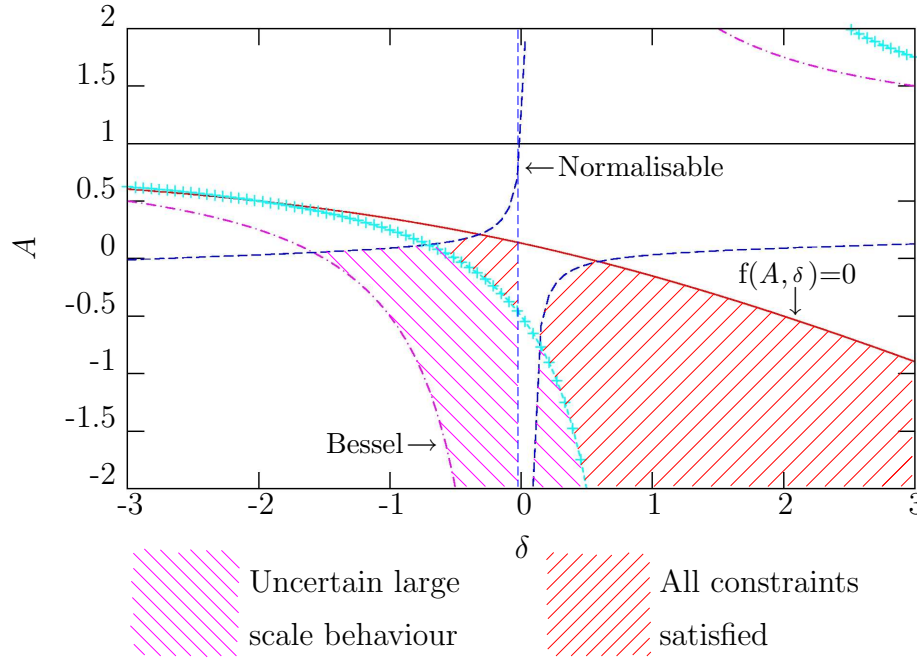


FIG. 2: The regions of parameter space in which the wave-functions of the self-adjoint Hamiltonian constraint equation with lattice refinement are physically acceptable. Notice that there are regions (crosses) in which the Taylor expansions used to calculate the large scale behaviour of the wave-functions are no longer valid, and hence whilst we can say that these wave-functions decay sufficiently quickly on large scales to be normalisable and are physical (i.e. oscillating), we cannot be sure that there is no new large scale behaviour due to the underlying discreteness.

corrections becoming significant at large scales provided

$$f(A, \delta_\nu) \equiv A^2 + \left( \frac{\delta_\nu}{2} - 2 \right) A + \frac{1}{4} - \frac{\delta_\nu}{2} \geq 0, \quad (5.8)$$

with further restrictions on  $\epsilon_\nu(\phi)$  for the case of equality [4]. This is shown in Fig. 2.

A similar calculation for the self-adjoint case gives

$$\lim_{\text{large } \nu} \Delta \nu_n = \frac{\pi}{\sqrt{\epsilon_\nu}} \nu^{\frac{(4-2\delta_\nu)(1-A)-3}{4(1-A)}} + \mathcal{O} \left( \nu^{\frac{4(1-\delta_\nu)(1-A)-6}{4(1-A)}} \right), \quad (5.9)$$

which is precisely the same equation we had for the non-self-adjoint case, Eq (5.3). In addition, the Taylor and Bessel expansions used are valid in the same ranges as those of the non-self-adjoint case. This is not surprising since making the constraint equation self-adjoint is inherently a quantum operation and their classical limits should be the same. Thus, Fig. 2 applies to self-adjoint, as well as to non-self-adjoint Hamiltonian operators, in the lattice refinement case (albeit with a different constraint coming from the requirement that the coefficients be normalisable).

By considering the underlying origins of lattice refinement, we can further restrict the allowed range to  $0 < A < -1/2$  [3]. This allows us to examine the types of matter that cannot be supported by a particular lattice refinement model. The relevant section of Fig. 2 is replotted in Fig. 3. Notice that Eq. (3.6) is needed to find the scaling behaviour of a particular matter component with respect to the scale factor, i.e. matter scaling like  $\nu^{\delta_\nu}$ ,

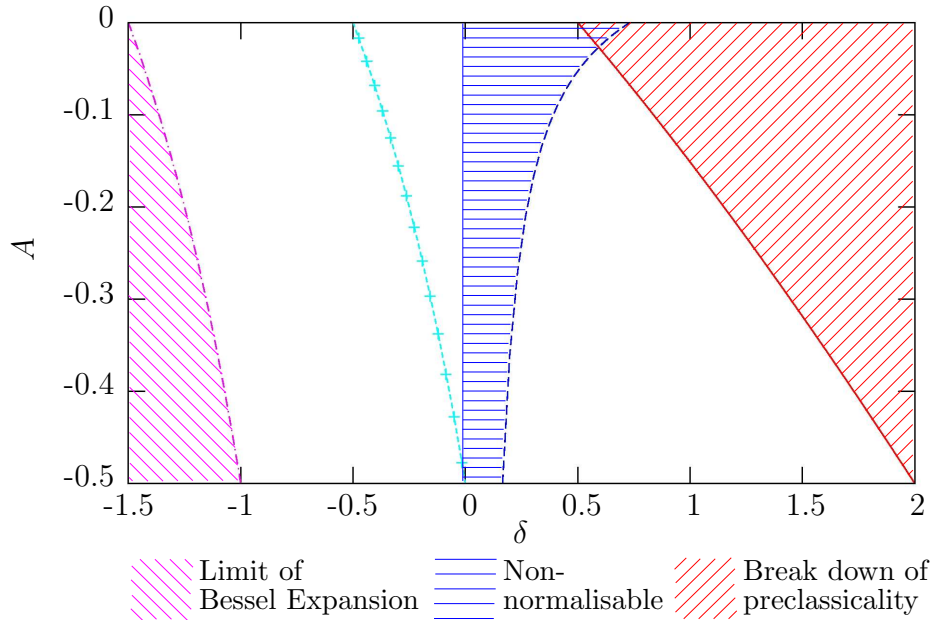


FIG. 3: The full loop quantum gravity theory allows only the range  $0 < A < -1/2$ . Within this range we can see that the acceptable types of matter content are significantly restricted. In addition, notice that for a varying lattice ( $A \neq 0$ ) it is not always possible to treat the large scale behaviour of the wave-functions perturbatively (dashed line with crosses).

scales with respect to the scale factor as  $a^{2\delta_\nu(1-A)}$ .

## VI. CONCLUSIONS

We have derived, in the continuum limit, the Hamiltonian constraint of loop quantum cosmology for a general lattice refinement scheme of the form  $\tilde{\mu} = \mu_0 \mu^A$ , for both the self-adjoint and non-self-adjoint Hamiltonian operator cases. We solved the resulting Wheeler-deWitt like equations and discussed the requirements the solutions must satisfy in order to be physically viable. These requirements give us constraints on the type of matter that can be supported by a particular lattice refinement model. We considered the following three requirements for the wave-functions: (i) that the coefficients of their basis expansion be normalisable, (ii) that they have oscillating large scale solutions so as to ensure that classical dynamics can be recovered and (iii) that they are pre-classical at large scales. Combining these conditions significantly constrains the allowable region of parameter space.

In particular, for the case of a constant lattice, physical wave-functions are produced only if  $\mathcal{H}_\phi$  scales faster than  $a^{-1}$  and slower than  $a$ ; an extremely severe restriction, given that many types of matter scale beyond this range. In the most popular lattice refinement model,  $A = -1/2$ , this range is extended so that physical wave-functions are produced provided  $\mathcal{H}_\phi$  scales faster than  $a^{-3}$  and slower than  $a^6$ , although it is not possible to treat the large scale oscillations perturbatively over a third of this range ( $a^{-3} \rightarrow a^0$ ).

As a concrete example, an inflationary scalar field (i.e., one in which the potential term dominates over the kinetic term in the matter Hamiltonian), scales like  $a^3$ . From our general procedure it is clear that this has a large scale breakdown of pre-classicality for the fixed lattice case, whilst this problem is resolved for the common,  $A = -1/2$ , lattice refinement

case, as was shown in [4]. This provides a further demonstration of the importance of modelling lattice refinement in loop quantum cosmology, if physical results are to be produced, and it does so for a large class of such models.

It is important to note that lattice refinement could, in principle, be much more complicated than the power law form ( $\tilde{\mu} = \mu_0 \mu^A$ ) used here, however even with this simplifying assumption the qualitative behaviour of different lattice refinement models is clear. In particular, we have shown that the continuum limit of the Hamiltonian constraint equation is sensitive to the choice of model and that only a limited range of matter components can be supported within a particular choice. This further emphasises the need to support effective, phenomenological lattice refinement models with a deeper understanding of the fundamental theory.

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